Midterm (Take-Home) + Solutions (Due 2/19/21)

Your Name:

Names of Any Collaborators:

This midterm exam is graded out of 27 points and is worth 25% of your overall score in the class. This take-home exam is due by 11:59PM on 2/19/21, submitted via http://gradescope.com. Your overall score on the midterm exam is worth 25% of your overall grade.

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

The simple rules for the exam are:

1. You may freely use any theorems or problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. You may freely consult the textbook or my handwritten lecture notes.
2. You cannot use any results that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work. Please name those with whom you have discussed the problems with.

I will pursue anyone suspected of breaking these rules.

You should **turn in this cover page** together with all of the work that you have decided to submit.

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

M1. (**7 points**) Let D be a domain and assume that $f : D \to \mathbb{C}$ is continuous. Determine which statements are true, and which are false. If you think a statement is true, *briefly* explain your reasoning. If you think a statement is false, you must prove it by providing a counterexample. Each statement is worth 1 point: awarded for correctly identifying the truth value (with justification, as described in the preceding two sentences). No points will be awarded if you fail to follow the directions.

(i) If f is analytic on D, then $\int_C f(z)\,dz = 0$ for any closed contour C lying in D.

Solution. This is **false**. The problem is that D may not be simply connected! For a counter example, consider $D = \mathbb{C} \setminus \{0\}$ (a domain that is not simply connected) and $f(z) = \frac{1}{z}$ (a function that is analytic on D) and let C be the positively oriented circle of radius 1 centered at 0. We have seen in the lecture that $\int_C \frac{1}{z}\,dz = 2\pi i$. ☕

(ii) If f has an antiderivative on D, then $\int_C f(z)\,dz = 0$ for any closed contour in D.

Solution. This is **true**. This is just one of the implications from the Fundamental Theorem of Contour Integrals. ☕

(iii) Suppose that $f : D \to \mathbb{C}$ is analytic on the domain D except for at a single singularity z_0. Let C_R be a positively oriented circle of radius $R > 0$ (small enough that the circle lies in D) centered at z_0. Then

$$\int_{C_R} f(z)\,dz = \lim_{R\to 0} \int_{C_R} f(z)\,dz.$$

Solution. This is **true**. Let $0 < r < R$. Then f is analytic on C_R, C_r and the domain consisting of the points that are both interior to C_R and exterior to C_r. By the Principle of Deformation of Paths, we can write

$$\int_{C_R} f(z)\,dz = \int_{C_r} f(z)\,dz.$$

Taking the limit as $r \to 0$ proves the claim. ☕

(iv) If f is analytic on D, then there exists an analytic function $F : D \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in D$.

Solution. This is **false**. Again, the problem is that D may not be simply connected. The function $f(z) = \frac{1}{z}$ on the domain $\mathbb{C} \setminus \{0\}$ is a counterexample again. We have already seen that $\int_C \frac{1}{z}\,dz = 2\pi i$ when C is a circle about 0. By the Fundamental Theorem of Contour Integrals, this implies that f has no antiderivative on D. ☕

(v) Let C be any circle with positive orientation and R the closed disk consisting of C and its interior. If f is entire and constant on C, then f is constant on R.

Solution. This is **true**. Suppose that for all $z \in \mathbb{C}$, $f(z) = w$ for some $w \in \mathbb{C}$. Let z_0 be any point interior to R. Since f is entire, it is analytic everywhere on R. Hence, by the

Cauchy Integral Theorem,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z_0}\,ds \\ &= \frac{k}{2\pi i}\int_C \frac{1}{s-z_0}\,ds \\ &= k. \end{aligned}$$

This proves that $f(z) = k$ for all $z \in R$. ☕

(vi) If $\int_C f(z)\,dz = 0$ for any closed contour C lying in D, then the real and imaginary parts of f satisfy the Cauchy-Riemann equations on D.

Solution. This is **true**. By Morera's theorem, f is analytic on D and hence satisfies the Cauchy-Riemann equations on D. ☕

(vii) If f is entire and $n \in \mathbb{N}$, then there exists an entire function F such that $F^{(n)}(z) = f(z)$ for all $z \in \mathbb{C}$ (here $F^{(n)}$ denotes the n-th derivative of F).

Solution. This is **true**. We proved in the lecture that every entire function has an antiderivative. That is, there exists an analytic function $F : \mathbb{C} \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in C$. But F itself is entire, and hence has an antiderivative. Proceed by induction to prove the claim. ☕

M2. (**4 points**) Let C be any positively oriented simple closed contour not crossing through the points i and $-i$. Determine (with proof) all possible values of the integral

$$\int_C \frac{e^{\pi z}}{z^2+1}\,dz.$$

Solution. The function has singularities at $z = i, -i$, so the integral may depend on which singularities are enclosed by the contour. Therefore, there are four cases to consider.

Case 1. Suppose that neither i nor $-i$ are interior to C. Then the integrand is analytic everywhere on and interior to C and hence

$$\int_C \frac{e^{\pi z}}{z^2+1}\,dz = 0$$

by Cauchy Goursat.

Case 2. Suppose that i is interior to C but $-i$ is not interior to C. Then the function $g(z) = \frac{e^{\pi z}}{z+i}$ is analytic on and interior to C. Hence, by the Cauchy Integral Theorem

$$\int_C \frac{e^{\pi z}}{z^2+1}\,dz = \int_C \frac{g(z)}{z-i}\,dz = 2\pi i g(i) = 2\pi i \frac{e^{\pi i}}{2i} = -\pi.$$

Case 3. Suppose that $-i$ is interior to C but i is not interior to C. Then the function $h(z) = \frac{e^{\pi z}}{z-i}$ is analytic on and interior to C. Hence, by the Cauchy Integral Theorem

$$\int_C \frac{e^{\pi z}}{z^2+1}\,dz = \int_C \frac{h(z)}{z+i}\,dz = 2\pi i h(-i) = 2\pi i \frac{e^{-\pi i}}{-2i} = \pi.$$

Case 4. Suppose that both i and $-i$ are interior to C. In this case, let C_i and C_{-i} denote positively oriented circles, centered at i and $-i$ respectively, with radii chosen so small that the circles lie interior to C and do not intersect eachother. Then by the Generalized Cauchy Goursat Theorem, we can write

$$\begin{aligned}\int_C \frac{e^{\pi z}}{z^2+1}\,dz &= \int_{C_i} \frac{e^{\pi z}}{z^2+1}\,dz + \int_{C_{-i}} \frac{e^{\pi z}}{z^2+1}\,dz \\ &= -\pi + \pi && \text{(by Case 2 and 3)} \\ &= 0.\end{aligned}$$

This completes the solution.

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M3. (**4 points**) Let C be a simple closed positively oriented contour whose interior contains -1. Compute the integral

$$\frac{1}{2\pi i}\int_C \frac{z^n}{(z+1)^{k+1}}\,dz$$

for any integers $0 \le k \le n$.

Solution. The function $f(z) = z^n$ is entire and hence analytic everywhere on and interior to C. Since -1 lies interior to C, we can apply the generalized Cauchy Integral Formula. The k-th derivative of f is given by

$$f^{(k)}(z) = n(n-1)(n-2)\cdots(n-k+1)z^{n-k}.$$

Hence, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_C \frac{z^n}{(z+1)^{k+1}}\,dz &= \frac{1}{k!}\left(\frac{k!}{2\pi i}\int_C \frac{z^n}{(z+1)^{k+1}}\right) \\ &= \frac{1}{k!}f^{(k)}(-1) && \text{(Cauchy Integral Formula)} \\ &= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}(-1)^{n-k} \\ &= \frac{n!}{k!(n-k)!}(-1)^{n-k} \\ &= \binom{n}{k}(-1)^{n-k},\end{aligned}$$

This completes the solution. ☕

M4. (**4 points**) Suppose that f is analytic everywhere on $\mathbb{C}$ except at a finite number of singular points $z_1, \ldots, z_n$ and let C be any positively oriented simple closed contour whose interior contains the singularities. Describe, in detail, how one can use the Generalized Cauchy-Goursat Theorem to compute the integral $\displaystyle\int_C f(z)\,dz$. You may assume that the value of a contour integral of f is known whenever the contour contains precisely one singularity of f in its interior.

Solution. Let $C_1, \ldots, C_n$ be positively oriented circles centered at $z_1, \ldots, z_n$. For each $i = 1, \ldots, n$, let R_i denote the region consisting of C_i and its interior. Choose the radius of each circle so small that the following properties are satisfied:

(1) The regions R_i are all contained in the interior of C;

(2) The regions R_i are pairwise disjoint, that is, $R_i \cap R_j = \emptyset$ if $i \neq j$.

Criteria (1) is possible because each point z_i is interior to C and criteria (2) is possible because there are only finitely many circles. Now, f is analytic everywhere on C except at $z_1, \ldots, z_n$. Hence, f is analytic on the region consisting of the contours $C, C_1, \ldots, C_n$ and the points that are both interior to C and exterior to each C_i. By the Generalized Cauchy Goursat Theorem, we can then write

$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz.$$

By assumption, each contour C_i contains precisely one singularity of f and hence by assumption we know the value of the integral $\int_{C_i} f(z)\,dz$.. This completes the solution. ☕

M5. (**4 points**) Suppose that f is an entire function satisfying $|f(z)| \leq \sqrt{1+|z|}$ for all $z \in \mathbb{C}$. Prove that f is constant.[1]

Proof. Let $R > 0$ and $z_0 \in \mathbb{C}$. Since f is entrie, it is analytic everywhere on and interior to the circle $C_R(z_0)$. By Cauchy's inequality, we obtain

$$\begin{aligned} |f'(z_0)| &\leq \frac{\max\limits_{z \in C_R(z_0)} |f(z)|}{R} \\ &\leq \frac{\max\limits_{z \in C_R(z_0)} \sqrt{1+|z|}}{R} && (|f(z)| \leq \sqrt{1+|z|}) \\ &\leq \frac{\max\limits_{z \in C_R(z_0)} \sqrt{1+|z-z_0|+|z_0|}}{R} && \text{(Triangle Inequality)} \\ &= \frac{\sqrt{1+R+|z_0|}}{R}. \end{aligned}$$

The right hand side converges to 0 as $R \to \infty$, which implies that $|f'(z_0)| = 0$ and hence $f'(z_0) = 0$. This proves that $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$. But $\mathbb{C}$ is a domain, so this implies that f is constant on $\mathbb{C}$. This proves the claim. ☕

M6. (**4 points**) Suppose that $f(z) = u(x,y) + iv(x,y)$ is an entire function and that there exists $M > 0$ such that $|u(x,y)| \leq M$ for all $z = x + iy \in \mathbb{C}$. Prove that f is constant.[2]

Proof. The function $g(z) = e^{f(z)}$ is a composition of entire functions and hence is entire by the chain rule. We will first show that $g(z)$ is bounded. Let $z \in \mathbb{C}$. Then we have

$$\begin{aligned} |g(z)| &= \left|e^{f(z)}\right| \\ &= \left|e^{u(x,y)}\right| \cdot \left|e^{iv(x,y)}\right| \\ &= e^{u(x,y)} \\ &\leq e^{M}. \end{aligned}$$

[1]Hint: Try to show $f' = 0$ using Cauchy's Inequality.
[2]Hint: try to apply Liouville's theorem to $e^{f(z)}$.

Hence, $g(z)$ is bounded on the complex plane. By Liouville, $g(z)$ is constant. Choose $w \in \mathbb{C}$ such that$e^{f(z)} = g(z) = w$ for $z \in \mathbb{C}$. Then by the chain rule,

$$f'(z)e^{f(z)} = g'(z) = 0.$$

By the zero-product property, we conclude that $f'(z) = 0$ for all $z \in \mathbb{C}$. But $\mathbb{C}$ is a domain so this implies that f is constant on $\mathbb{C}$. This completes the proof. ☕